Order Statistics and Probabilistic Robust Control*

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Abstract

Order statistics theory is applied in this paper to probabilistic robust control theory to compute the minimum sample size needed to come up with a reliable estimate of an uncertain quantity under continuity assumption of the related probability distribution. Also, the concept of distribution-free tolerance intervals is applied to estimate the range of an uncertain quantity and extract the information about its distribution. To overcome the limitations imposed by the continuity assumption in the existing order statistics theory, we have derived a cumulative distribution function of the order statistics without the continuity assumption and developed an inequality showing that this distribution has an upper bound which equals to the corresponding distribution when the continuity assumption is satisfied. By applying this inequality, we investigate the minimum computational effort needed to come up with an reliable estimate for the upper bound (or lower bound) and the range of a quantity. We also give conditions, which are much weaker than the absolute continuity assumption, for the existence of such minimum sample size. Furthermore, the issue of making tradeoff between performance level and risk is addressed and a guideline for making this kind of tradeoff is established. This guideline can be applied in general without continuity assumption.

Keywords: Order Statistics, Probabilistic Robustness, Minimum Sample Size, Distribution Inequality, Design Tradeoff, Tolerance Intervals

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1 Introduction

It is now well-known that many deterministic worst-case robust analysis and synthesis problems are NP hard, which means that the exact analysis and synthesis of the corresponding robust control problems may be computational demanding [6, 13, 15]. Moreover, the deterministic worst-case robustness measures may be quite conservative due to overbounding of the system uncertainties. As pointed out by Khargonekar and Tikku in [9], the difficulties of deterministic worst-case robust control problems are inherent to the problem formulations and a major change of the paradigm is necessary. An alternative to the deterministic approach is the probabilistic approach which has been studied extensively by Stengel and co-workers, see for example, [10], [11], and references therein. Aimed at breaking through the NP-hardness barrier and reducing the conservativeness of the deterministic robustness measures, the probabilistic approach has recently received a renewed attention in the work by Barmish and Lagoa [4], Barmish, Lagoa, and Tempo [5], Barmish and Polyak [3], Khargonekar and Tikku [9], Bai, Tempo, and Fu [1], Tempo, Bai, and Dabbene [12], Yoon and Khargonekar [14], and references therein.

In particular, Tempo, Bai and Dabbene in [12] and Khargonekar and Tikku in [9] have derived bounds for the number of samples required to estimate the upper bound of a quantity with a certain a priori specified accuracy and confidence. It is further shown that this probabilistic approach for robust control analysis and synthesis has low complexity [9, 12]. It should also be pointed out that the uncertain parameters do not necessarily have to be random, they can be regarded as randomized variables as pointed out in [9].

One important open question in this area is the minimum computational effort (i.e., the minimum sample size) needed to obtain a reliable estimate for the upper bound (or lower bound, or the range) of an uncertain quantity. We shall answer this question in this paper. This paper is organized as follows. Section 2 presents the problem formulation and motivations. In Section 3, we derive a distribution inequality without the continuity assumption. Design tradeoff is discussed in Section 4 using the distribution derived in Section 3. Section 5 gives the minimum sample size under various assumptions and Section 6 considers the tolerance intervals.

2 Problem Formulation and Motivations

Let q be a random vector, bounded in a compact set $\mathcal{Q}$, with a multivariate probability density function $\varpi(q)$. Let $u(q)$ be a real scalar measurable function of the random vector q with cumulative probability distribution $F_u(\gamma) := P_{rob}\{u(q) \leq \gamma\}$ for a given $\gamma \in \mathbf{R}$. Let $q^1, q^2, \cdots, q^N$ be the i.i.d. (independent and identically distributed) samples of q generated according to the same probability density function $\varpi(q)$ where N is the sample size. Now define random variable $\hat{u}_i,\ i = 1, 2, \cdots, N$

as the $i-$th smallest observation of $u(q)$ during N sample experiments. These random variables are called *order statistics* [2, 7] because $\hat{u}_1 \leq \hat{u}_2 \leq \hat{u}_3 \leq \cdots\cdots \leq \hat{u}_N$.

We are interested in computing the following probabilities:

- $P_{rob}\left\{P_{rob}\left\{u(q) > \hat{u}_n\right\} \leq \varepsilon\right\}$ for any $1 \leq n \leq N$, and $\varepsilon \in (0,1)$;
- $P_{rob}\left\{P_{rob}\left\{u(q) < \hat{u}_m\right\} \leq \varepsilon\right\}$ for any $1 \leq m \leq N$, and $\varepsilon \in (0,1)$;
- $P_{rob}\left\{P_{rob}\left\{\hat{u}_m < u(q) \leq \hat{u}_n\right\} \geq 1-\varepsilon\right\}$ for any $1 \leq m < n \leq N$, and $\varepsilon \in (0,1)$.

In the subsequent subsections, we shall give some motivations for computing such probabilities.

2.1 Robust Analysis and Optimal Synthesis

As noted in [9] and [12], to tackle robust analysis or optimal synthesis problem, it is essential to deal with the following questions:

- What is $\max_{\mathcal{Q}} u(q)$ (or $\min_{\mathcal{Q}} u(q)$)?
- What is the value of q at which $u(q)$ achieves $\min_{\mathcal{Q}} u(q)$ (or $\max_{\mathcal{Q}} u(q)$)?

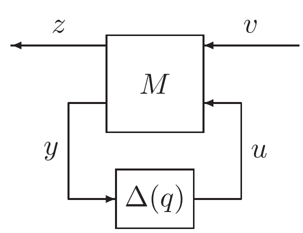

Figure 1: Uncertain System

Consider, for example, an uncertain system shown in Figure 1. Denote the transfer function from v to z by T_{zv} and suppose that T_{zv} has the following state space realization $T_{zv} = \left[\begin{array}{c|c} A(q) & B(q) \\ \hline C(q) & D(q) \end{array}\right]$. We can now consider following robustness problems:

- Robust stability: Let $u(q) := \max_i \mathrm{Re}\,\lambda_i(A(q))$ where $\lambda_i(A)$ denotes the i-th eigenvalue of A. Then the system is robustly stable if $\max_{q\in\mathcal{Q}} u(q) < 0$.
- Robust performance: Suppose $A(q)$ is stable for all $q \in \mathcal{Q}$. Define $u(q) := ||T_{zv}||_\infty$. Then the robust performance problem is to determine if $\max_{q\in\mathcal{Q}} u(q) \leq \gamma$ is satisfied for some prespecified $\gamma > 0$.

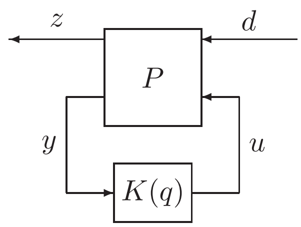

Figure 2: Synthesis Framework

As another example, consider a dynamical system shown in Figure 2 and suppose q is a vector of controller parameters to be designed. Denote the transfer function from d to z by T_{zd} and suppose T_{zd} has the following state space realization $T_{zd} = \left[\begin{array}{c|c} A_s(q) & B_s(q) \\ \hline C_s(q) & D_s(q) \end{array}\right]$. Let $u(q) := ||T_{zd}||_\infty$ and $\mathcal{Q}_s = \{q \in \mathcal{Q} : \; A_s(q) \text{ is stable}\}$. Then an H_∞ optimal design problem is to determine a vector of parameters q achieving $\min_{\mathcal{Q}_s} \; u(q)$.

In general, exactly evaluating $\min_{\mathcal{Q}} \; u(q)$ (or $\max_{\mathcal{Q}} \; u(q)$) or determining q achieving it may be an NP hard problem and thus is intractable in practice. Henceforth, we adopt the probabilistic approach proposed in [9] and [12]. That is, estimating $\min_{\mathcal{Q}} u(q)$ as

$$\hat{u}_1 = \min_{i \in \{1,2,\cdots,N\}} \; \hat{u}_i$$

for sufficiently large N and computing $P_{rob} \left\{ P_{rob} \left\{ u(q) < \hat{u}_1 \right\} \leq \varepsilon \right\}$ for a small $\varepsilon \in (0,1)$ to see how reliable the estimation is. Similarly, we estimate $\max_{\mathcal{Q}} u(q)$ as $\hat{u}_N = \max_{i \in \{1,2,\cdots,N\}} \; \hat{u}_i$ for sufficiently large N and consider $P_{rob} \left\{ P_{rob} \left\{ u(q) > \hat{u}_N \right\} \leq \varepsilon \right\}$.

2.2 Quantity Range

In many applications, estimating only the upper bound (or lower bound) for a quantity is not sufficient. It is also important to estimate the range of the quantity with a certain accuracy and confidence level. For example, in pole placement problem, we need to know the range which the poles fall into. Suppose that q is the vector of uncertain parameters or design parameters of a system and $u(q)$ is an uncertain quantity, for example, $u(q)$ may be the H_∞ norm of a closed-loop transfer function or the maximum real part of the eigenvalues of the closed-loop system matrix. Intuitively, the range of quantity $u(q)$ can be approached by $(\hat{u}_1, \hat{u}_N]$ as sample size N goes to infinity. Therefore, it is important to know $P_{rob} \left\{ P_{rob} \left\{ \hat{u}_1 < u(q) \leq \hat{u}_N \right\} \geq 1 - \varepsilon \right\}$.

So far, we have only concerned the lower bound and (or) upper bound of uncertain quantity $u(q)$. Actually, it is desirable to know its distribution function $F_u(.)$. This is because $F_u(.)$ contains all the information of the quantity. However, the exact computation of $F_u(.)$ is, in general, intractable [16]. An alternative is to extract as much as possible the information of the distribution function $F_u(.)$ from the observations $\hat{u}_i, \; i = 1, 2, \cdots, N$. For this purpose, we are interested in computing

the probabilities asked at the beginning of this section. In particular, we will see in section 4 that, computing $P_{rob}\left\{P_{rob}\left\{u(q) > \hat{u}_n\right\} \leq \varepsilon\right\}$ is of great importance to make the tradeoff between the performance gradation and risk when designing a controller.

3 Distribution Inequality

Note that $P_{rob}\left\{u(q) \leq \hat{u}_i\right\} = F_u(\hat{u}_i), \quad i = 1, 2, \cdots, N$. To compute the probabilities asked at the beginning of section 2, it is important to know the associated distribution of any k random variables $F_u(\hat{u}_{i_1}),\ F_u(\hat{u}_{i_2}),\ \cdots,\ F_u(\hat{u}_{i_k}),\ \ 1 \leq i_1 < i_2 < \cdots < i_k \leq N,\ \ 1 \leq k \leq N$, i.e.,

$$F(t_1, t_2, \cdots, t_k) := P_{rob}\left\{F_u(\hat{u}_{i_1}) \leq t_1,\ F_u(\hat{u}_{i_2}) \leq t_2,\ \cdots,\ F_u(\hat{u}_{i_k}) \leq t_k\right\}.$$

To that end, we have the following theorem which follows essentially by combining *Probability Integral Transformation Theorem* in [8] and Theorem 2.2.3 in [7].

Theorem 1 *Let* $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq 1$. *Define*

$$f_{i_1, i_2, \cdots, i_k}(x_1, x_2, \cdots, x_k) := \prod_{s=0}^{s=k} N!\ \frac{(x_{s+1} - x_s)^{i_{s+1} - i_s - 1}}{(i_{s+1} - i_s - 1)!}$$

with $x_0 := 0,\ x_{k+1} := 1,\ i_0 := 0,\ i_{k+1} := N + 1$. *Suppose the cumulative distribution function* $F_u(\gamma) := P_{rob}\left\{u(q) \leq \gamma\right\}$ *is continuous. Then*

$$F(t_1, t_2, \cdots, t_k) = \int_{\mathbf{D}_{t_1, t_2, \cdots, t_k}} f_{i_1, i_2, \cdots, i_k}(x_1, x_2, \cdots, x_k)\ dx_1 dx_2 \cdots dx_k$$

where $\mathbf{D}_{t_1, t_2, \cdots, t_k} := \left\{(x_1, x_2, \cdots, x_k) :\ \ 0 \leq x_1 \leq x_2 \leq \cdots \leq x_k,\ x_s \leq t_s,\ s = 1, 2, \cdots, k\right\}.$

Remark 1 *Theorem 1 can play an important role in robust control as illustrated in the following sections. However, its further application is limited by the continuity assumption. In many robust control problems, it is reasonable to assume that $u(q)$ is measurable, while the continuity of $F_u(\gamma)$ is not necessarily guaranteed. For example, $F_u(\gamma)$ is not continuous when uncertain quantity $u(q)$ equals to a constant in an open set of $\mathcal{Q}$. We can come up with many uncertain systems with which the continuity assumption for the distribution of quantity $u(q)$ is not guaranteed. To tackle these problems without continuity assumption by probabilistic approach and investigate the minimum computational effort, we shall develop a distribution inequality which accommodates the case when the continuity is not guaranteed.*

First, we shall established the following lemma.

Lemma 1 *Let U be a random variable with uniform distribution over* $[0, 1]$ *and* $\hat{U}_n, \ \ n = 1, 2, \cdots, N$ *be the order statistics of U, i.e.,* $\hat{U}_1 \leq \hat{U}_2 \leq \cdots \leq \hat{U}_N$. *Let* $0 = t_0 < t_1 < t_2 < \cdots < t_k \leq 1$.

Define $G_{j_1,j_2,\cdots,j_k}(t_1,t_2,\cdots,t_k) := (1-t_k)^{N-\sum_{l=1}^{k} j_l} \prod_{s=1}^{k} \binom{N-\sum_{l=1}^{s-1} j_l}{j_s} (t_s - t_{s-1})^{j_s}$ *and* $\mathbf{I}_{i_1,i_2,\cdots,i_k} := \{(j_1,j_2,\cdots,j_k) : \ i_s \leq \sum_{l=1}^{s} j_l \leq N, \ \ s=1,2,\cdots,k\}$. *Then*

$$P_{rob}\left\{\hat{U}_{i_1} \leq t_1, \hat{U}_{i_2} \leq t_2, \cdots, \hat{U}_{i_k} \leq t_k\right\} = \sum_{(j_1,j_2,\cdots,j_k)\in \mathbf{I}_{i_1,i_2,\cdots,i_k}} G_{j_1,j_2,\cdots,j_k}(t_1,t_2,\cdots,t_k)$$

Proof. Let j_s be the number of samples of U which fall into $(t_{s-1}, t_s]$, $\ s = 1,2,3,\cdots,k$. Then the number of samples of U which fall into $[0, t_s]$ is $\sum_{l=1}^{s} j_l$. It is easy to see that the event $\left\{\hat{U}_{i_s} \leq t_s\right\}$ is equivalent to event $\{i_s \leq \sum_{l=1}^{s} j_l \leq N\}$. Furthermore, the event

$$\left\{\hat{U}_{i_1} \leq t_1, \hat{U}_{i_2} \leq t_2, \cdots, \hat{U}_{i_k} \leq t_k\right\}$$

is equivalent to the event $\{i_s \leq \sum_{l=1}^{s} j_l \leq N, \ \ s=1,2,\cdots,k\}$. Therefore,

$$\begin{aligned}
& P_{rob}\left\{\hat{U}_{i_1} \leq t_1, \hat{U}_{i_2} \leq t_2, \cdots, \hat{U}_{i_k} \leq t_k\right\} \\
= & \sum_{(j_1,j_2,\cdots,j_k)\in \mathbf{I}_{i_1,i_2,\cdots,i_k}} \prod_{s=1}^{k} \binom{N-\sum_{l=1}^{s-1} j_l}{j_s} (t_s - t_{s-1})^{j_s} \ (1-t_k)^{N-\sum_{l=1}^{k} j_l} \\
= & \sum_{(j_1,j_2,\cdots,j_k)\in \mathbf{I}_{i_1,i_2,\cdots,i_k}} G_{j_1,j_2,\cdots,j_k}(t_1,t_2,\cdots,t_k).
\end{aligned}$$

□

Theorem 2 *Let* $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq 1$. *Define*

$$\tilde{F}(t_1,t_2,\cdots,t_k) := P_{rob}\left\{F_u(\hat{u}_{i_1}) < t_1, \ F_u(\hat{u}_{i_2}) < t_2, \ \cdots, \ F_u(\hat{u}_{i_k}) < t_k\right\}$$

and $\tau_s := \sup_{\{x:\, F_u(x)<t_s\}} F_u(x)$, $\ s=1,2,\cdots,k$. *Suppose* $u(q)$ *is a measurable function of* q. *Then*

$$\tilde{F}(t_1,t_2,\cdots,t_k) = \int_{\mathbf{D}_{\tau_1,\tau_2,\cdots,\tau_k}} f_{i_1,i_2,\cdots,i_k}(x_1,x_2,\cdots,x_k)\, dx_1 dx_2 \cdots dx_k.$$

Furthermore, $\tilde{F}(t_1,t_2,\cdots,t_k) \leq \int_{\mathbf{D}_{t_1,t_2,\cdots,t_k}} f_{i_1,i_2,\cdots,i_k}(x_1,x_2,\cdots,x_k)\, dx_1 dx_2 \cdots dx_k$ *and the equality holds if* $F_u(\gamma)$ *is continuous.*

Proof. Define $\alpha_0 := -\infty$ and $\alpha_s := \sup\{x : \ F_u(x) < t_s\}$, $\ \alpha_s^- := \alpha_s - \epsilon$, $\ s = 1,2,\cdots,k$ where $\epsilon > 0$ can be arbitrary small. Let $\phi_s := F_u(\alpha_s^-)$, $\ s=1,2,\cdots,k$. We can show that $\phi_l < \phi_s$ if $\alpha_l < \alpha_s$, $\ 1 \leq l < s \leq k$. In fact , if this is not true, we have $\phi_l = \phi_s$. Because ϵ can be arbitrary small, we have $\alpha_s^- \in (\alpha_l, \ \alpha_s)$. Notice that $\alpha_l = \min\{x : \ F_u(x) \geq t_l\}$, we have $t_l \leq \phi_s = \phi_l$. On the other hand, by definition we know that $\alpha_l^- \in \{x : \ F_u(x) < t_l\}$ and thus $\phi_l = F_u(\alpha_l^-) < t_l$, which is a contradiction. Notice that $F_u(\gamma)$ is nondecreasing and right-continuous,

we have $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$ and $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq \phi_k \leq 1$ and that event $\{F_u(\hat{u}_{i_s}) < t_s\}$ is equivalent to the event $\{\hat{u}_{i_s} < \alpha_s\}$. Furthermore, event $\{F_u(\hat{u}_{i_1}) < t_1,\ F_u(\hat{u}_{i_2}) < t_2,\ \cdots,\ F_u(\hat{u}_{i_k}) < t_k\}$ is equivalent to event $\{\hat{u}_{i_1} < \alpha_1,\ \hat{u}_{i_2} < \alpha_2,\ \cdots,\ \hat{u}_{i_k} < \alpha_k\}$ which is defined by k constraints $\hat{u}_{i_s} < \alpha_s,\ s = 1, 2, \cdots, k$. For every $l < k$, delete constraint $\hat{u}_{i_l} < \alpha_l$ if there exists $s > l$ such that $\alpha_s = \alpha_l$. Let the remained constraints be $\hat{u}_{i'_s} < \alpha'_s,\ s = 1, 2, \cdots, k'$ where $\alpha'_1 < \alpha'_2 < \cdots < \alpha'_{k'}$. Since all constraints deleted are actually redundant, it follows that event $\{\hat{u}_{i_1} < \alpha_1,\ \hat{u}_{i_2} < \alpha_2,\ \cdots,\ \hat{u}_{i_k} < \alpha_k\}$ is equivalent to event $\left\{\hat{u}_{i'_1} < \alpha'_1,\ \hat{u}_{i'_2} < \alpha'_2,\ \cdots,\ \hat{u}_{i'_{k'}} < \alpha'_{k'}\right\}$. Now let j_s be the number of observations of $u(q)$ which fall into $[\alpha'_{s-1}, \alpha'_s),\quad s = 1, 2, \cdots, k'$. Then the number of observations of $u(q)$ which fall into $(-\infty, \alpha'_s)$ is $\sum_{l=1}^{s} j_l$. It is easy to see that the event $\left\{\hat{u}_{i'_s} < \alpha'_s\right\}$ is equivalent to the event $\left\{i'_s \leq \sum_{l=1}^{s} j_l \leq N\right\}$. Furthermore, the event $\left\{\hat{u}_{i'_1} < \alpha'_1,\ \hat{u}_{i'_2} < \alpha'_2,\ \cdots,\ \hat{u}_{i'_{k'}} < \alpha'_{k'}\right\}$ is equivalent to event $\left\{i'_s \leq \sum_{l=1}^{s} j_l \leq N,\quad s = 1, 2, \cdots, k'\right\}$. Therefore

$$
\begin{aligned}
& \tilde{F}(t_1, t_2, \cdots, t_k) = P_{rob}\{F_u(\hat{u}_{i_1}) < t_1,\ F_u(\hat{u}_{i_2}) < t_2,\ \cdots,\ F_u(\hat{u}_{i_k}) < t_k\} \\
= \;& P_{rob}\{\hat{u}_{i_1} < \alpha_1,\ \hat{u}_{i_2} < \alpha_2,\ \cdots,\ \hat{u}_{i_k} < \alpha_k\} = P_{rob}\left\{\hat{u}_{i'_1} < \alpha'_1,\ \hat{u}_{i'_2} < \alpha'_2,\ \cdots,\ \hat{u}_{i'_{k'}} < \alpha'_{k'}\right\} \\
= \;& \sum_{(j_1, j_2, \cdots, j_{k'}) \in \mathbf{I}_{i'_1, i'_2, \cdots, i'_{k'}}} \prod_{s=1}^{k'} \binom{N - \sum_{l=1}^{s-1} j_l}{j_s} [F_u(\alpha'^{-}_s) - F_u(\alpha'^{-}_{s-1})]^{j_s} \, [1 - F_u(\alpha'^{-}_{k'})]^{N - \sum_{l=1}^{k'} j_l} \\
= \;& \sum_{(j_1, j_2, \cdots, j_{k'}) \in \mathbf{I}_{i'_1, i'_2, \cdots, i'_{k'}}} G_{j_1, j_2, \cdots, j_{k'}}\left(\phi'_1, \phi'_2, \cdots, \phi'_{k'}\right).
\end{aligned}
$$

Now consider event $\left\{\hat{U}_{i_1} \leq \phi_1,\ \hat{U}_{i_2} \leq \phi_2,\ \cdots,\ \hat{U}_{i_k} \leq \phi_k\right\}$. For every $l < k$, delete constraint $\hat{U}_{i_l} \leq \phi_l$ if there exists $s > l$ such that $\phi_s = \phi_l$. Notice that $\phi_s = F_u(\alpha_s^-)$ and $\phi_l < \phi_s$ if $\alpha_l < \alpha_s,\ 1 \leq l < s \leq k$, the remained constraints must be be $\hat{U}_{i'_s} \leq \phi'_s,\ s = 1, 2, \cdots, k'$ where $\phi'_s = F_u(\alpha'^{-}_s),\ s = 1, 2, \cdots, k'$ and $\phi'_1 < \phi'_2 < \cdots < \phi'_{k'}$. Since all constraints deleted are actually redundant, it follows that event $\left\{\hat{U}_{i_1} \leq \phi_1,\ \hat{U}_{i_2} \leq \phi_2,\ \cdots,\ \hat{U}_{i_k} \leq \phi_k\right\}$ is equivalent to event $\left\{\hat{U}_{i'_1} \leq \phi'_1,\ \hat{U}_{i'_2} \leq \phi'_2,\ \cdots,\ \hat{U}_{i'_{k'}} \leq \phi'_{k'}\right\}$. By Theorem 2.2.3 in [7] and Lemma 1

$$
\begin{aligned}
& \int_{\mathbf{D}_{\phi_1, \phi_2, \cdots, \phi_k}} f_{i_1, i_2, \cdots, i_k}(x_1, x_2, \cdots, x_k)\, dx_1 dx_2 \cdots dx_k \\
= \;& P_{rob}\left\{\hat{U}_{i_1} \leq \phi_1,\ \hat{U}_{i_2} \leq \phi_2,\ \cdots,\ \hat{U}_{i_k} \leq \phi_k\right\} = P_{rob}\left\{\hat{U}_{i'_1} \leq \phi'_1,\ \hat{U}_{i'_2} \leq \phi'_2,\ \cdots,\ \hat{U}_{i'_{k'}} \leq \phi'_{k'}\right\} \\
= \;& \sum_{(j_1, j_2, \cdots, j_{k'}) \in \mathbf{I}_{i'_1, i'_2, \cdots, i'_{k'}}} G_{j_1, j_2, \cdots, j_{k'}}\left(\phi'_1, \phi'_2, \cdots, \phi'_{k'}\right).
\end{aligned}
$$

Therefore, $\tilde{F}(t_1, t_2, \cdots, t_k) = \int_{\mathbf{D}_{\phi_1, \phi_2, \cdots, \phi_k}} f_{i_1, i_2, \cdots, i_k}(x_1, x_2, \cdots, x_k)\, dx_1 dx_2 \cdots dx_k$. By the definitions of τ_s and ϕ_s, we know that $\mathbf{D}_{\tau_1, \tau_2, \cdots, \tau_k}$ is the closure of $\mathbf{D}_{\phi_1, \phi_2, \cdots, \phi_k}$, i.e., $\mathbf{D}_{\tau_1, \tau_2, \cdots, \tau_k} = \bar{\mathbf{D}}_{\phi_1, \phi_2, \cdots, \phi_k}$

and that their Lebesgue measures are equal. It follows that

$$\tilde{F}(t_1, t_2, \cdots, t_k) = \int_{\mathbf{D}_{\tau_1,\tau_2,\cdots,\tau_k}} f_{i_1,i_2,\cdots,i_k}(x_1, x_2, \cdots, x_k)\; dx_1 dx_2 \cdots dx_k.$$

Notice that $\tau_s \leq t_s, \;\; s = 1, 2, \cdots, k$, we have $\mathbf{D}_{\tau_1,\tau_2,\cdots,\tau_k} \subseteq \mathbf{D}_{t_1,t_2,\cdots,t_k}$ and hence

$$\tilde{F}(t_1, t_2, \cdots, t_k) \leq \int_{\mathbf{D}_{t_1,t_2,\cdots,t_k}} f_{i_1,i_2,\cdots,i_k}(x_1, x_2, \cdots, x_k)\; dx_1 dx_2 \cdots dx_k.$$

Furthermore, if $F_u(\gamma)$ is continuous, then $\tau_s = t_s, \;\; s = 1, 2, \cdots, k$, hence $\mathbf{D}_{\tau_1,\tau_2,\cdots,\tau_k} = \mathbf{D}_{t_1,t_2,\cdots,t_k}$ and the equality holds. $\square$

4 Performance and Confidence Tradeoff

For the synthesis of a controller for an uncertain system, we usually have a conflict between the performance level and robustness. The following theorem helps to make the tradeoff.

Theorem 3 *Let $1 \leq n \leq N, \; 1 \leq m \; \leq N$ and $\varepsilon \in (0, 1)$. Suppose $u(q)$ is measurable. Then*

$$P_{rob}\left\{P_{rob}\left\{u(q) > \hat{u}_n\right\} \leq \varepsilon\right\} \geq 1 - \int_0^{1-\varepsilon} \frac{N!}{(n-1)!(N-n)!} x^{n-1}(1-x)^{N-n} dx \tag{1}$$

and the equality holds if and only if $\sup_{\{x:\, F_u(x) < 1-\varepsilon\}} F_u(x) = 1 - \varepsilon$; Moreover,

$$P_{rob}\left\{P_{rob}\left\{u(q) < \hat{u}_m\right\} \leq \varepsilon\right\} \geq 1 - \int_0^{1-\varepsilon} \frac{N!}{(m-1)!(N-m)!} x^{N-m}(1-x)^{m-1} dx \tag{2}$$

and the equality holds if and only if $\inf_{\{x:\, F_u(x) > \varepsilon\}} F_u(x) = \varepsilon$.

Proof. Apply Theorem 2 to the case of $k = 1, \; i_1 = n$, we have

$$\begin{aligned} & P_{rob}\left\{F_u(\hat{u}_n) < 1 - \varepsilon\right\} \\ = \;& \int_{\mathbf{D}_\tau} \frac{N!}{(n-1)!(N-n)!} x^{n-1}(1-x)^{N-n} dx \leq \int_0^{1-\varepsilon} \frac{N!}{(n-1)!(N-n)!} x^{n-1}(1-x)^{N-n} dx \end{aligned}$$

where $\mathbf{D}_\tau = (0, \tau]$ with $\tau = \sup_{\{x:\, F_u(x) < 1-\varepsilon\}} F_u(x)$. Therefore,

$$\begin{aligned} & P_{rob}\left\{P_{rob}\left\{u(q) > \hat{u}_n\right\} \leq \varepsilon\right\} = P_{rob}\left\{F_u(\hat{u}_n) \geq 1 - \varepsilon\right\} \\ = \;& 1 - P_{rob}\left\{F_u(\hat{u}_n) < 1 - \varepsilon\right\} \geq 1 - \int_0^{1-\varepsilon} \frac{N!}{(n-1)!(N-n)!} x^{n-1}(1-x)^{N-n} dx. \end{aligned}$$

The equality holds if and only if $\sup_{\{x:\, F_u(x) < 1-\varepsilon\}} F_u(x) = 1 - \varepsilon$ because $\mathbf{D}_\tau = (0, 1 - \varepsilon]$ if and only if $\tau = \sup_{\{x:\, F_u(x) < 1-\varepsilon\}} F_u(x) = 1 - \varepsilon$.

Now let $v(q) = -u(q)$. Let the cumulative distribution function of $v(q)$ be $F_v(.)$ and define order statistics $\hat{v}_i, \;\; i = 1, 2 \cdots, N$ as the i-th smallest observation of $v(q)$ during N i.i.d. sample

experiments, i.e., $\hat{v}_1 \leq \hat{v}_2 \leq \hat{v}_3 \leq \cdots\cdots \leq \hat{v}_N$. Obviously, $\hat{u}_m = -\hat{v}_{N+1-m}$ for any $1 \leq m \leq N$. It is also clear that $F_v(-x) = 1 - F_u(x^-)$, which leads to the equivalence of $\sup_{\{x:\, F_v(x)<1-\varepsilon\}} F_v(x) = 1-\varepsilon$ and $\inf_{\{x:\, F_u(x)>\varepsilon\}} F_u(x) = \varepsilon$. Therefore, apply (1) to the situation of $v(q)$, we have

$$\begin{aligned} & P_{rob}\left\{P_{rob}\left\{u(q) < \hat{u}_m\right\} \leq \varepsilon\right\} \\ = \;& P_{rob}\left\{P_{rob}\left\{v(q) > \hat{v}_{N+1-m}\right\} \leq \varepsilon\right\} \geq 1 - \int_0^{1-\varepsilon} \frac{N!}{(m-1)!(N-m)!} x^{N-m}(1-x)^{m-1} dx \end{aligned}$$

and the equality holds if and only if $\inf_{\{x:\, F_u(x)>\varepsilon\}} F_u(x) = \varepsilon$. □

In Figure 3 we computed the lower bound for $P_{rob}\left\{P_{rob}\left\{u(q) > \hat{u}_n\right\} \leq \varepsilon\right\}$ for sample size $N = 8000$ with $\varepsilon = 0.0010$, $\varepsilon = 0.0012$, and $\varepsilon = 0.0015$ respectively.

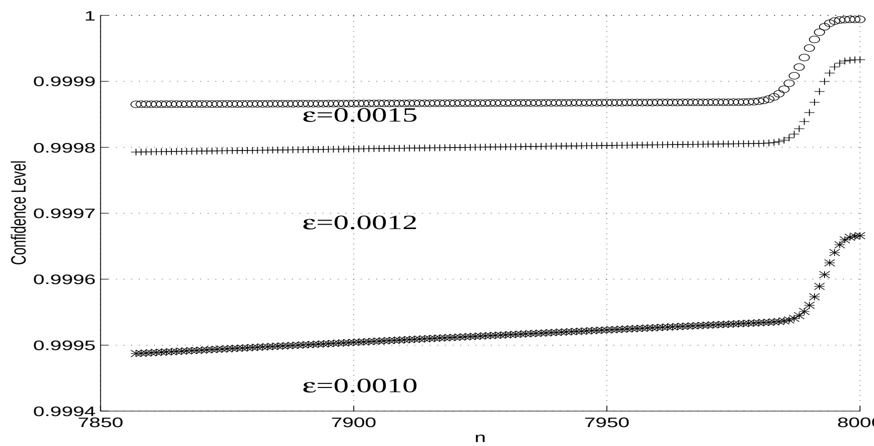

Figure 3: Confidence Level := $P_{rob}\left\{P_{rob}\left\{u(q) > \hat{u}_n\right\} \leq \varepsilon\right\}$. Sample size $N = 8000$. Performance level increases as n decreases, while confidence level decreases.

Theorem 3 can be used as a guideline to robust control analysis and synthesis. For example, when dealing with robust stability problem, we need to compute the maximum of the real part of the closed-loop poles, denoted by $u(q)$, which is a function of uncertain parameters q. When we estimate the upper bound of $u(q)$ by sampling, it is possible that most of the samples concentrated in an interval and very few fall far beyond that interval. If we take $\hat{u}_N$ as the maximum in the design of a controller, it may be conservative. However, if we choose n such that $\hat{u}_n$ is much smaller than $\hat{u}_N$, while $P_{rob}\left\{P_{rob}\left\{u(q) > \hat{u}_n\right\} \leq \varepsilon\right\}$ is close to 1, then the controller based on $\hat{u}_n$ may have much better performance but with only a little bit more increase of risk. For example, let's say, sample size $N = 8000$ and $\varepsilon = 0.0010$, the distribution of sample is like this, $\hat{u}_1,\ \hat{u}_2,\ \cdots,\ \hat{u}_{7900}$ concentrated in an interval, and $\hat{u}_{7900}$ is much smaller than $\hat{u}_{7901},\ \hat{u}_{7902},\ \cdots, \hat{u}_{8000}$. It is sure that the controller designed by taking $\hat{u}_{7900}$ as the upper bound will have much higher performance level than the controller designed by taking $\hat{u}_{8000}$ as the upper bound. To compare the risks for these two cases, we have $P_{rob}\left\{P_{rob}\left\{u(q) > \hat{u}_{8000}\right\} \leq 0.0010\right\} \geq 0.99966$ and $P_{rob}\left\{P_{rob}\left\{u(q) > \hat{u}_{7900}\right\} \leq 0.0010\right\} \geq 0.99951$.

These data indicate that there is only a little bit more increase of risk by taking $\hat{u}_{7900}$ instead of $\hat{u}_{8000}$ as the upper bound in designing a controller.

5 Minimum Sample Size

In addition to the situation of making the tradeoff between performance degradation and risk, Theorem 3 can also play an important role in the issue of computational effort required to come up with an estimate of the upper bound (or lower bound) of a quantity with a certain accuracy and confidence. This issue was first addressed independently by Tempo, Bai and Dabbene in [12] and Khargonekar and Tikku in [9] and their results are summarized in the following theorem.

Theorem 4 *For any* $\varepsilon, \delta \in (0,1)$, $P_{rob}\{P_{rob}\{u(q) > \hat{u}_N\} \leq \varepsilon\} \geq 1-\delta$ *if* $N \geq \frac{\ln \frac{1}{\delta}}{\ln \frac{1}{1-\varepsilon}}$.

Remark 2 *This theorem only answers the question that how much computational effort is sufficient. We shall also concern about what is the minimum computational effort. By applying Theorem 3 to the case of* $n = N$*, we can also obtain Theorem 4. Moreover, we can see that, for a certain accuracy (i.e., a fixed value of* ε*), the bound becomes minimum if and only if*

$$\sup_{\{x:\, F_u(x) < 1-\varepsilon\}} F_u(x) = 1-\varepsilon. \tag{3}$$

If $F(u)$ *is continuous (i.e., (3) is guaranteed for any* $\varepsilon \in (0,1)$*), the bound is of course tight.*

Similarly, by Theorem 3 to the case of $m = 1$ *we have that for* $\varepsilon, \delta \in (0,1)$,

$$P_{rob}\{P_{rob}\{u(q) \geq \hat{u}_1\} \geq 1-\varepsilon\} \geq 1-\delta$$

if $N \geq \frac{\ln \frac{1}{\delta}}{\ln \frac{1}{1-\varepsilon}}$*. For a fixed* $\varepsilon \in (0,1)$*, this bound is tight if and only if* $\inf_{\{x:\, F_u(x) > \varepsilon\}} F_u(x) = \varepsilon$.

6 Quantity Range and Distribution-Free Tolerance Intervals

To estimate the range of an uncertain quantity with a certain accuracy and confidence level apriori specified, we have the following corollary.

Corollary 1 *Suppose* $F_u(\gamma)$ *is continuous. For any* $\varepsilon, \delta \in (0,1)$,

$$P_{rob}\{P_{rob}\{\hat{u}_1 < u(q) \leq \hat{u}_N\} \geq 1-\varepsilon\} \geq 1-\delta$$

if and only if $\mu(N) \leq \delta$ *where* $\mu(N) := (1-\varepsilon)^{N-1}[1+(N-1)\varepsilon]$ *is a monotonically decreasing function of* N.

Remark 3 *The minimum N guaranteeing this condition can be found by a simple bisection search. This bound of sample size is minimum because our computation of probability is exact. This bound is also practically small, for example, $N \geq 1,483$ if $\varepsilon = \delta = 0.005$, and $N \geq 9,230$ if $\varepsilon = \delta = 0.001$. Therefore, to obtain a reliable estimate of the range of an uncertain quantity, computational complexity is not an issue.*

In general, it is important to know the probability of a quantity falling between two arbitrary samples. To that end, we have

Corollary 2 *Suppose $F_u(\gamma)$ is continuous. Then for $\varepsilon \in (0,1)$ and $1 \leq m < n \leq N$,*

$$P_{rob}\left\{P_{rob}\left\{\hat{u}_m < u(q) \leq \hat{u}_n\right\} \geq 1-\varepsilon\right\} = \int_{1-\varepsilon}^{1} N\binom{N-1}{n-m-1} x^{n-m-1}(1-x)^{N-n+m}dx.$$

Here $(\hat{u}_m,\ \hat{u}_n]$ is referred as distribution-free tolerance interval in order statistics theory (see [7]).

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